Interação e Concorrência 2016/17
Bloco de slides 2

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Looking for suitable notions of equivalence of behaviours

Intuition

Two LTS should be equivalent if they cannot be distinguished by interacting with them.

Graph isomorphism

is too strong

Trace

Definition

Let $T = \langle S, N, \downarrow, s, \rightarrow \rangle$ be a labelled transition system. The set of traces $Tr(s)$, for $s \in S$ is the minimal set satisfying

(1) $\epsilon \in Tr(s)$

(2) $\checkmark \in Tr(s) \;\Leftrightarrow\; s \in \downarrow$

(3) $a\sigma \in Tr(s) \;\Rightarrow\; \langle \exists s' : s' \in S : \; s \xrightarrow{a} s' \wedge \sigma \in Tr(s') \rangle$

Trace equivalence

on states:

Two states s, r are trace equivalent iff $Tr(s) = Tr(r)$
(i.e. if they can perform the same finite sequences of transitions)

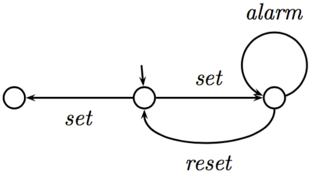
on LTSs:

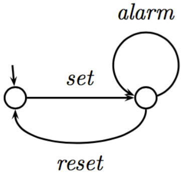
The LTS $T_1 = \langle S_1, N_1, \downarrow_1, s_1 \rightarrow_1 \rangle$ and $T_2 = \langle S_2, N_2, \downarrow_2, s_2 \rightarrow_2 \rangle$ are trace equivalent if

$$Tr(s_1) = Tr(s_2)$$

Trace equivalence

Example





Simulation

the quest for a behavioural equality:
able to identify states that cannot be distinguished by any realistic form of observation

Simulation

A state q simulates another state p if every transition from q is corresponded by a transition from p and this capacity is kept along the whole life of the system to which state space q belongs to.

Simulation

Definition

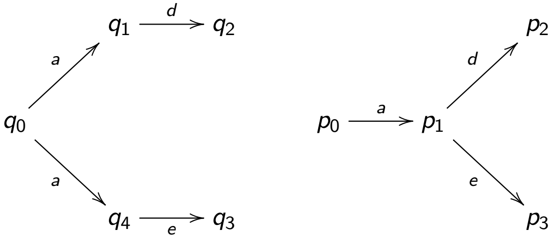
Given $\langle S_1, N, \downarrow_1, \rightarrow_1\rangle$ and $\langle S_2, N, \downarrow_2, \rightarrow_2\rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a simulation iff, for all $\langle p, q\rangle \in R$ and $a \in N$,

(1) $p \downarrow_1 \Rightarrow \; q \downarrow_2$

(2) $p \xrightarrow{a}_1 p' \;\Rightarrow\; \langle \exists q' \,:\, q' \in S_2 : q \xrightarrow{a}_2 \; q' \,\wedge\, \langle p', q'\rangle \in R\rangle$

$$\begin{array}{ccccccc} p & R & q & \qquad\Rightarrow\qquad & & & q \\ \downarrow{\scriptstyle a} & & & & & & \downarrow{\scriptstyle a} \\ p' & & & & p' & R & q' \end{array}$$

Example



$q_0 \lesssim p_0$ cf. $\{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$

Similarity

Definition
Smilarity

$$p \lesssim q \;\equiv\; \langle \exists R : R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

Lemma
The similarity relation is a preorder
(ie, reflexive and transitive)

Bisimulation

Definition (Bisimulation)

Given $\langle S_1, N, \downarrow_1, \rightarrow_1 \rangle$ and $\langle S_2, N, \downarrow_2, \rightarrow_2 \rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a bisimulation iff both R and its converse R° are simulations. I.e., whenever $\langle p, q \rangle \in R$ and $a \in N$,

$$\textbf{(1)} \quad p \downarrow_1 \Leftrightarrow \ q \downarrow_2$$

$$\textbf{(Zig)} \quad p \xrightarrow{a}_1 p' \ \Rightarrow \ \langle \exists\ q' \ : \ q' \in S_2 : \ q \xrightarrow{a}_2 q' \ \wedge \ \langle p', q' \rangle \in R \rangle$$

$$\textbf{(Zag)} \quad q \xrightarrow{a}_2 q' \ \Rightarrow \ \langle \exists\ p' \ : \ p' \in S_1 : \ p \xrightarrow{a}_1 p' \ \wedge \ \langle p', q' \rangle \in R \rangle$$

The Game characterization

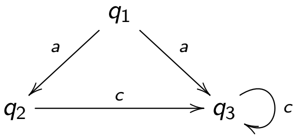
Two players R and I discuss whether the transition structures are mutually corresponding

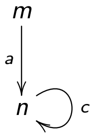
- R starts by chosing a transition
- I replies trying to match it
- if I succeeds, R plays again
- R wins if I fails to find a corresponding match
- I wins if it replies to all moves from R and the game is in a configuration where all states have been visited or R can't move further. In this case is said that I has a **wining strategy**

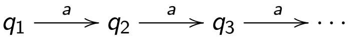
$s \sim t$ iff

I has an **universal wining strategy** from (s, t), i.e.,

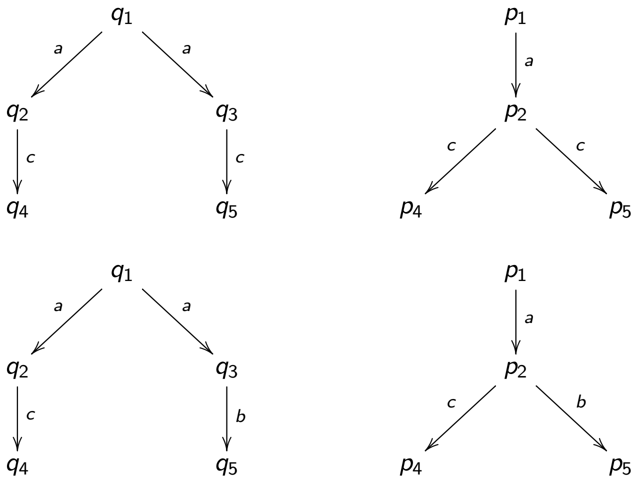
Examples







Examples



q_1
a
a
q_2
q_3
c
c
q_4
q_5
p_1
a
p_2
c
c
p_4
p_5
q_1
a
a
q_2
q_3
c
b
q_4
q_5
p_1
a
p_2
c
b
p_4
p_5

Bisimilarity

Definition (Bisimilarity)

$$p \sim q \;\equiv\; \langle \exists\ R\ \ ::\ \ R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

Lemma

1. The identity relation *id* is a bisimulation
2. The empty relation $\perp$ is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \,|\, i \in I\}$ is a bisimulation
6. $\sim$ is a bisimulation

Properties

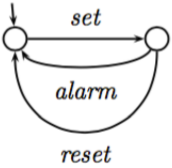
Lemma

The bisimilarity relation is an equivalence relation
(ie, reflexive, symmetric and transitive)

Lemma

The class of all bisimulations between two LTS has the structure of a complete lattice, ordered by set inclusion, whose top is the bisimilarity relation $\sim$.

Exercise



Define an LTS trace equivalent to the presented one, but with a distinct behaviour.

Properties

Lemma

In a **deterministic** labelled transition system, two states are bisimilar iff they are trace equivalent, i.e.,

$$s \sim s' \quad \Leftrightarrow \quad Tr(s) = Tr(s')$$

Hint: define a relation R as

$$\langle x, y \rangle \in R \;\Leftrightarrow\; Tr(x) = Tr(y)$$

and show R is a bisimulation.

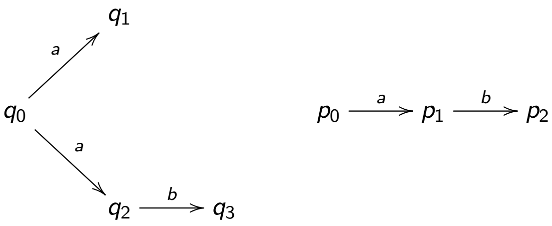
Properties

Warning

The bisimilarity relation $\sim$ is not the symmetric closure of $\lesssim$

Example

$$q_0 \lesssim p_0,\ p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$



Notes

Similarity as the greatest simulation

$$\lesssim \; \equiv \bigcup \{ S | S \text{ is a simulation} \}$$

Bisimilarity as the greatest bisimulation

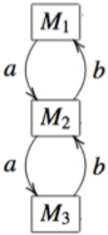
$$\sim \; \equiv \bigcup \{ S | S \text{ is a bisimulation} \}$$

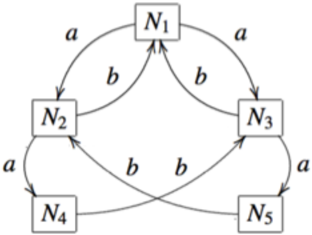
Exercises

Suppose a labelled transition system is given by the following transition relation:
$\{(1, a, 2), (1, a, 3), (2, a, 3), (2, b, 1), (3, a, 3), (3, b, 1),$
$(4, a, 5), (5, a, 5), (5, b, 6), (6, a, 5), (7, a, 8), (8, a, 8), (8, b, 7)\}$
Prove or refute $1 \sim 4 \sim 6 \sim 7$.

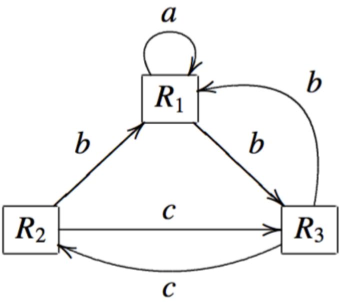
Exercises

Prove that $M_1 \sim N_1$:





Find an LTS with two states in a bisimulation relation with the states of the following LTS:



Exercises

Prove or refute the following sentences:

- “bisimulations are closed by unions”
- “bisimulations are closed by intersections”

Exercises

Given two labelled transition systems $\langle S_A, N, \downarrow_A, \rightarrow_A \rangle$ and $\langle S_B, N, \downarrow_B, \rightarrow_B \rangle$, two states p and q are *equisimilar* iff

$$p \doteqdot q \;\equiv\; p \lesssim q \wedge q \lesssim p$$

1. Show that $\doteqdot$ is an equivalence relation.
2. Compare this equivalence with bisimilarity $\sim$.

Exercises

A relation R over the state space of a labelled transition system is a *word bisimulation* if, whenever $\langle p, q \rangle \in R$ and $\sigma \in N^*$, we have

$$p \xrightarrow{\sigma} p' \;\Rightarrow\; \langle \exists q' : q' \in S_2 \;:\; q \xrightarrow{\sigma} q' \wedge \langle p', q' \rangle \in R \rangle$$
$$q \xrightarrow{\sigma} q' \;\Rightarrow\; \langle \exists p' : p' \in S_1 \;:\; p \xrightarrow{\sigma} p' \wedge \langle p', q' \rangle \in R \rangle$$

1. Define formally relation $\xrightarrow{\sigma}$, for $\sigma \in N^*$
2. Two states are *word bisimilar* iff they belong to a word bisimulation. Show that two states p and q are word bisimilar iff $p \sim q$.