Semantics of Higher-Order Probabilistic Programs with Continuous Distributions

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Based on work with or by Chris Heunen, Ohad Kammar, Sam Staton, and Frank Wood

Learning outcome

- Can explain what one can do with higher-order probabilistic programming language.
- Can use measure theory to interpret prob. prog. with continuous distributions.
- Can use quasi-Borel space to interpret higher-order prob. prog. with conditioning.

References

1. A convenient category for higher-order probability theory. Heunen et a. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.

What is probabilistic programming?

(Bayesian) probabilistic modelling of data

1. Develop a new probabilistic (generative) model.
2. Design an inference algorithm for the model.
3. Using the algo., fit the model to the data.

(Bayesian) probabilistic modelling of data in a prob. prog. language

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2. Design an inference algorithm for the model.
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(Bayesian) probabilistic modelling of data

in a prob. prog. language

as a program

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(Bayesian) probabilistic modelling of data

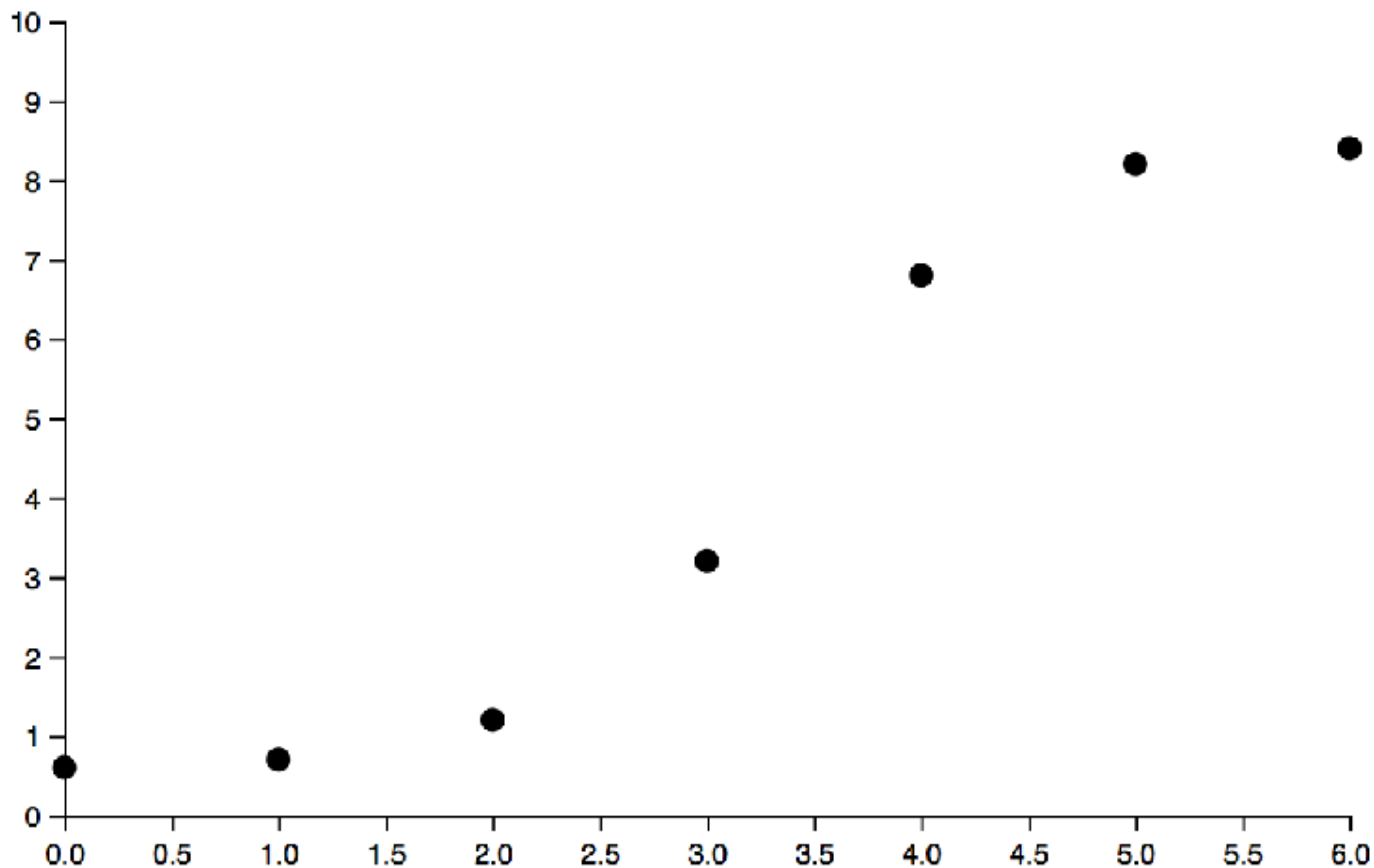
in a prob. prog. language

as a program

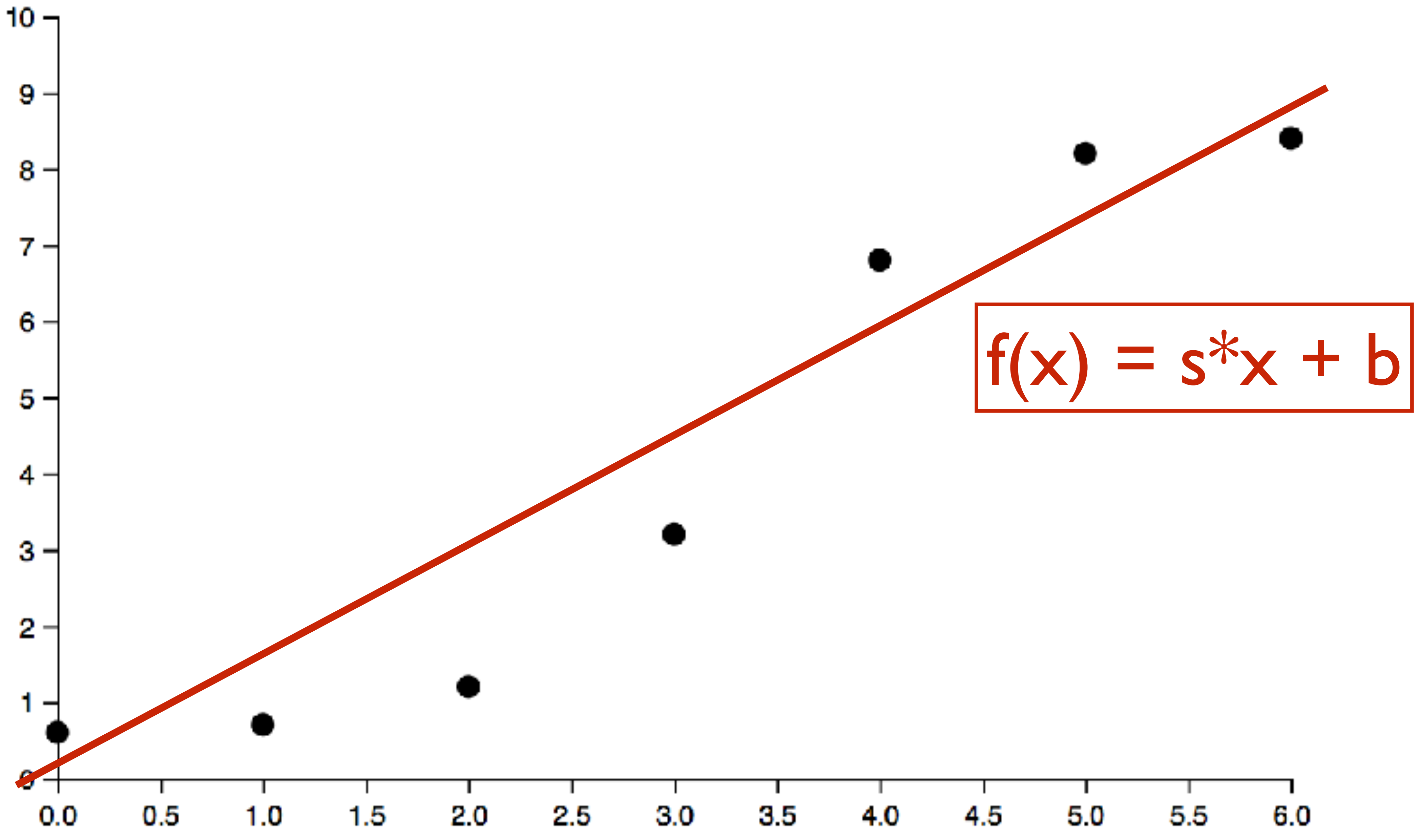
1. Develop a new probabilistic (generative) model.
2. ~~Design an inference algorithm for the model.~~
3. Using ~~the algo.~~, fit the model to the data.

a generic inference algo. of the language

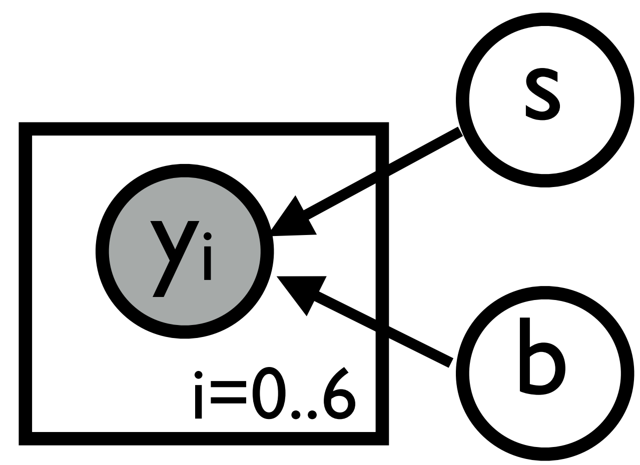
Line fitting



Line fitting



Bayesian generative model



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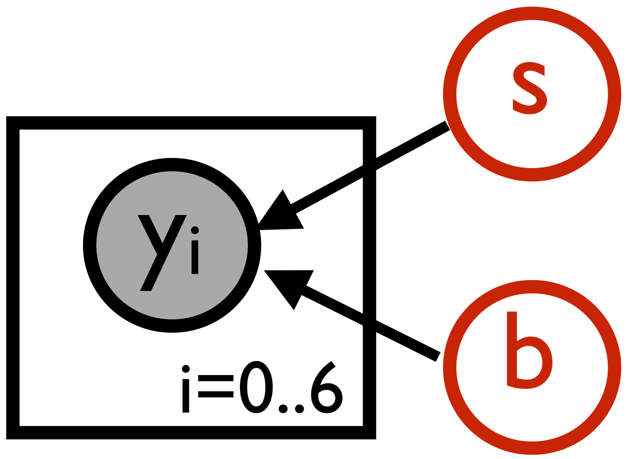
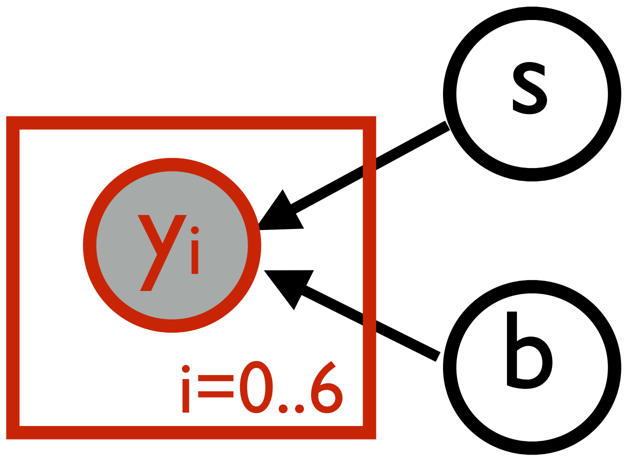


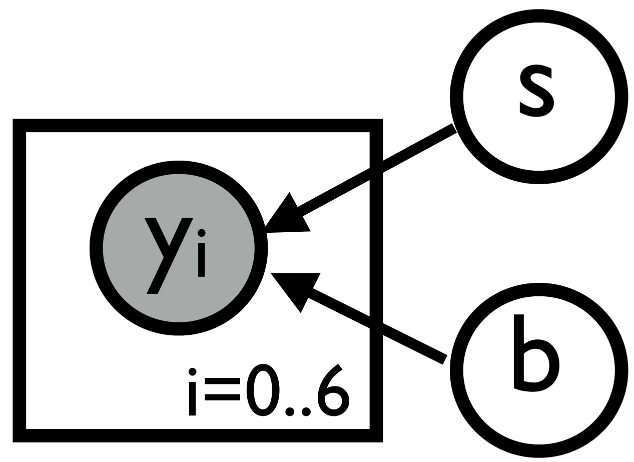

```
s      ~ normal(0, 2)
b      ~ normal(0, 6)
```

Bayesian generative model




```
s     ~ normal(0, 2)
b     ~ normal(0, 6)
f(x)  = s*x + b
yi     ~ normal(f(i), 0.5)
              where i = 0 .. 6
```

Bayesian generative model

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          where i = 0 .. 6
```

Q: posterior of (s,b) given y_0=0.6, …, y_6=8.4?

Posterior of s and b given y_i's

$$P(s, b \mid y_0, .., y_6) = \frac{P(y_0, .., y_6 \mid s,b) \times P(s,b)}{P(y_0, .., y_6)}$$

Posterior of s and b given y_i's

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(almost) Anglican program

```
(let [s (sample (normal 0 2))
      b (sample (normal 0 6))
      f (fn [x] (+ (* s x) b))]
```

(almost) Anglican program

```
(let [s (sample (normal 0 2))
      b (sample (normal 0 6))
      f (fn [x] (+ (* s x) b))]

    (observe (normal (f 0) .5) .6)
    (observe (normal (f 1) .5) .7)
    (observe (normal (f 2) .5) 1.2)
    (observe (normal (f 3) .5) 3.2)
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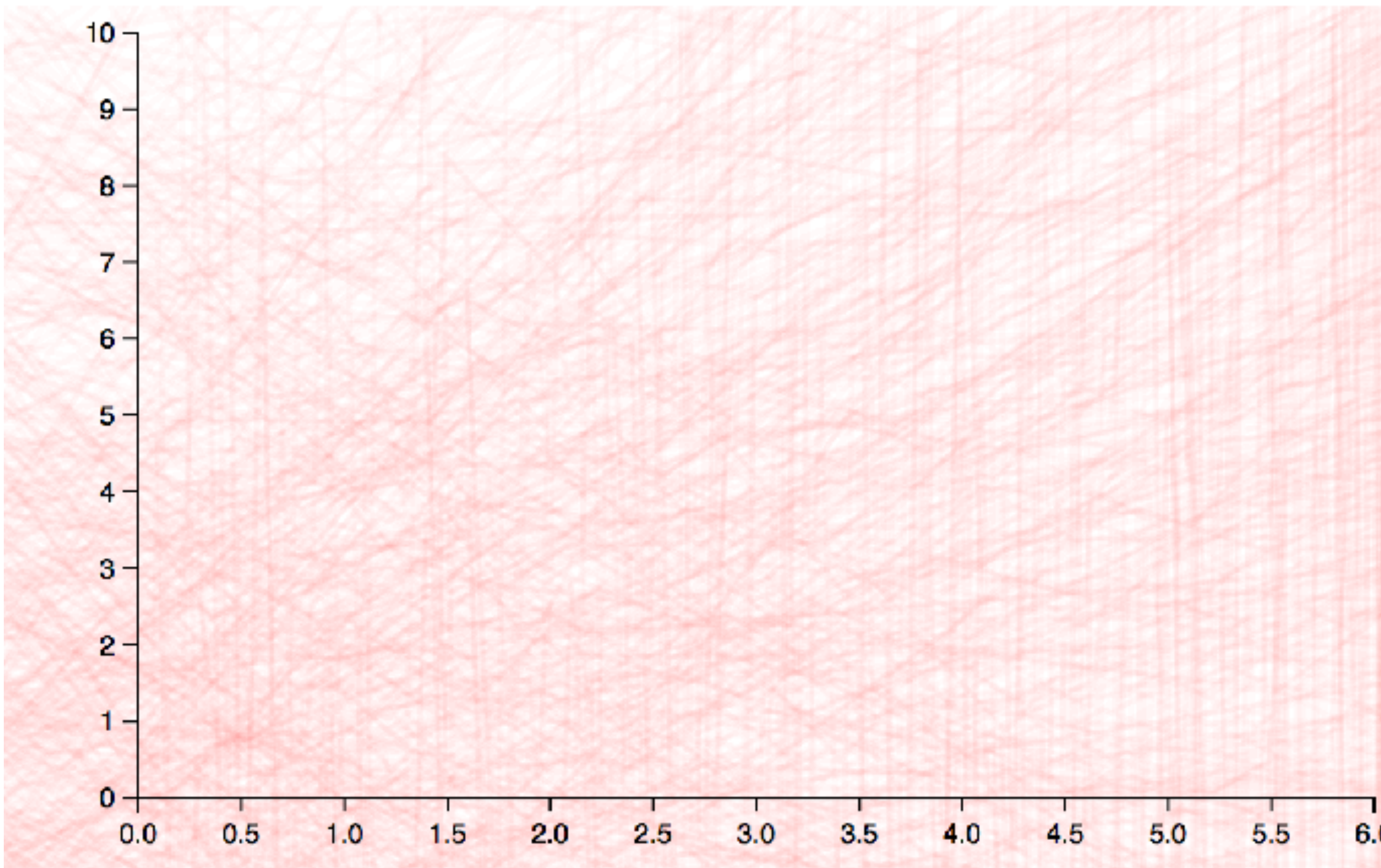
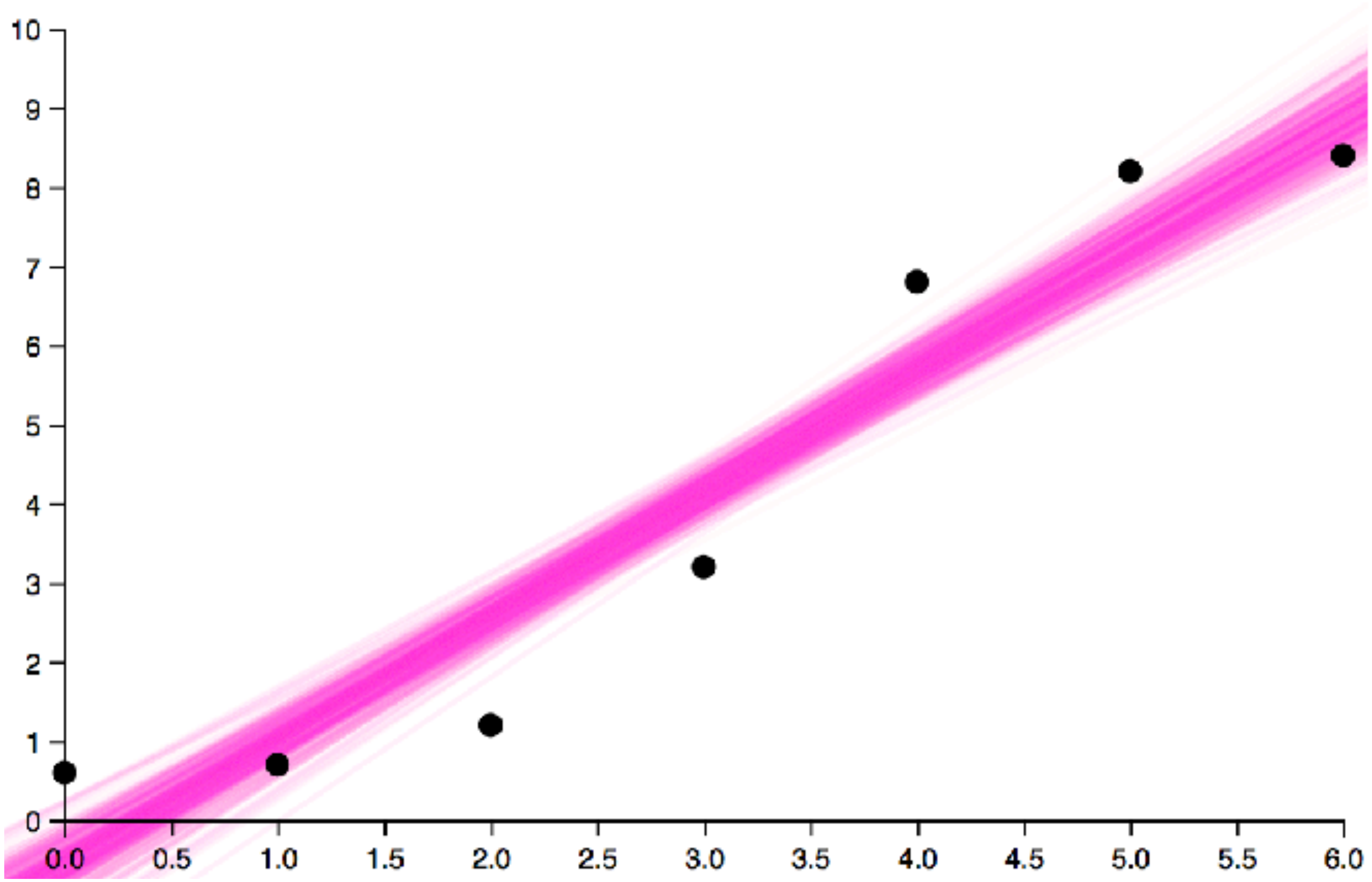
  [s b])
```

NB: (predict :sb [s b]) should be used instead of [s b] in Anglican

Samples from prior



Samples from posterior



Semantic challenges

```
(let [s (sample (normal 0 2))
      b (sample (normal 0 6))
      f (fn [x] (+ (* s x) b))]

     (observe (normal (f 0) .5) .6)
     (observe (normal (f 1) .5) .7)
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  [s b])
```

1. Continuous distributions.

Challenge 1: Continuous distributions

- Need care for handling continuous distributions on $\mathbb{R}$, to avoid paradoxes.
- Something like measure theory needed.
- Complex math.

```
(let [s (sample (normal 0 2))
      b (sample (normal 0 6))
      f (fn [x] (+ (* s x) b))]

    (observe (normal (f 0) .5) .6)
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  [s b])
```

1. Continuous distributions.
2. Higher-order functions.

```
(let [s (sample (normal 0 2))
      b (sample (normal 0 6))
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```

~~[s b])~~
f)

1. Continuous distributions.
2. Higher-order functions.

```
(let [F (fn []
          (let [s (sample (normal 0 2))
                b (sample (normal 0 6))]
            (fn [x] (+ (* s x) b))))
      f (F)]
    (observe (normal (f 0) .5) .6)
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    (observe (normal (f 2) .5) 1.2)
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~~[s b])~~
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1. Continuous distributions.
2. Higher-order functions.

```
(let [F (fn []
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            (fn [x] (+ (* s x) b))))
      f (add-change-points F 0 6)]
    (observe (normal (f 0) .5) .6)
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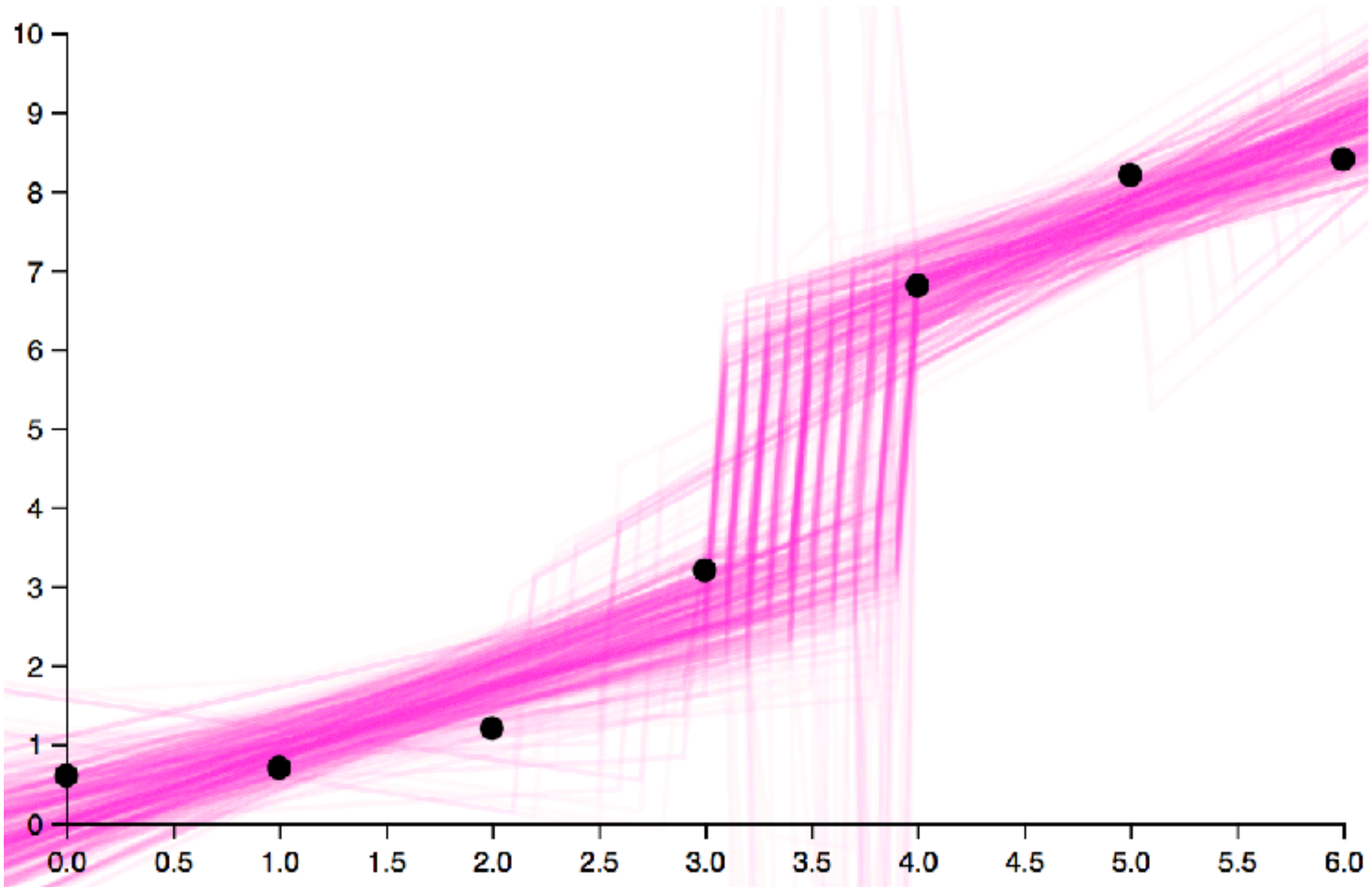
  [s b])
  f)
```

1. Continuous distributions.
2. Higher-order functions.

Samples from posterior



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~~[s b])~~

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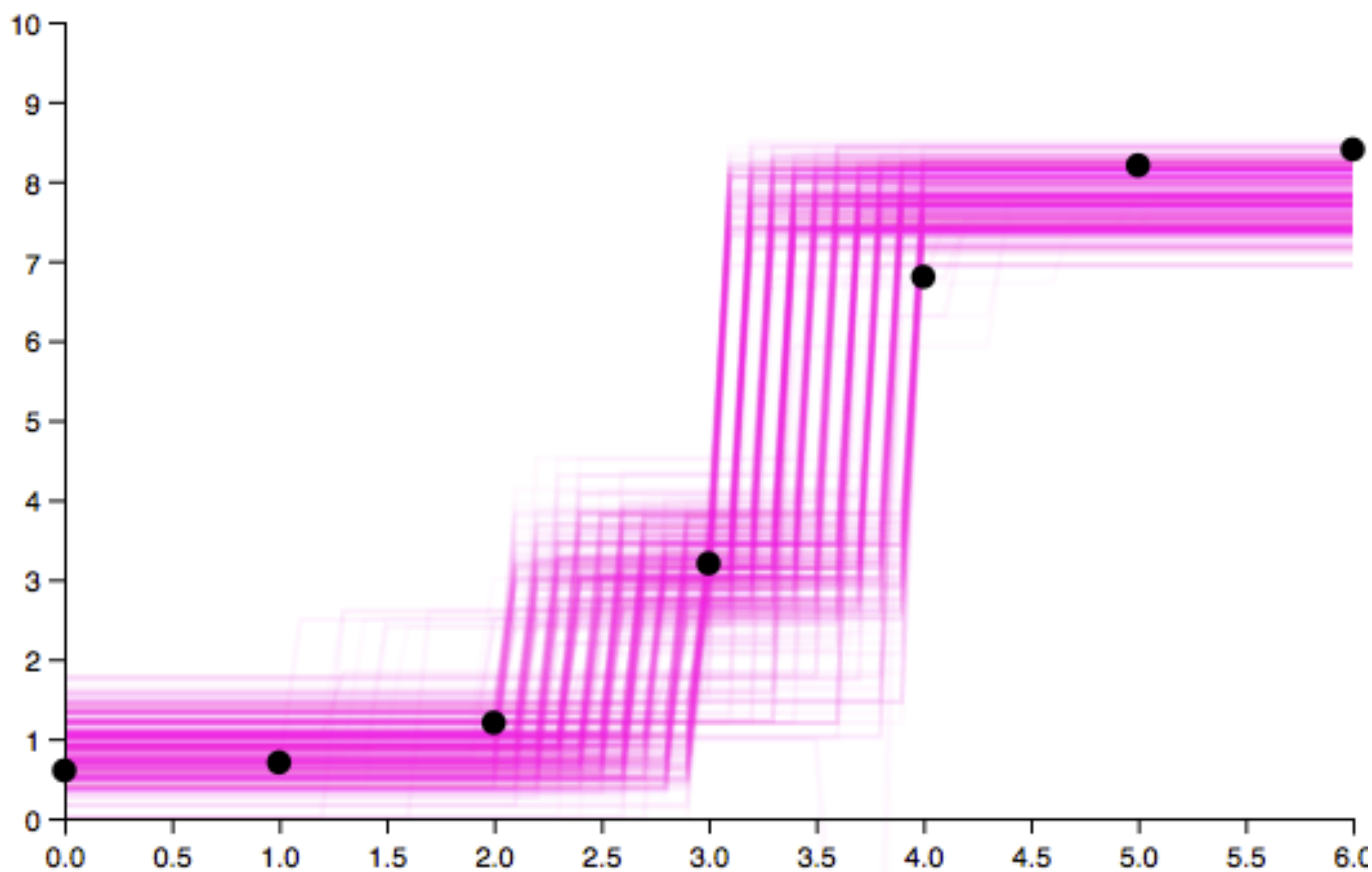

Challenge 2: Higher-order functions

Measure theory doesn't support HO fns well.

$$\mathrm{ev} : (\mathbb{R} \to_m \mathbb{R}) \times \mathbb{R} \to \mathbb{R}, \qquad \mathrm{ev}(f,x) = f(x).$$

[Aumann 61] ev is not measurable no matter which σ-algebra is used for $\mathbb{R} \to_m \mathbb{R}$.

[Cor] The category of measurable spaces is not cartesian closed.

```
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~~`[s b])`~~
`f)`

1. Continuous distributions.
2. Higher-order functions.

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  [s b])
  f)
```

1. Continuous distributions.
2. Higher-order functions.
3. Conditioning and prog. eqs.

Challenge 3: Conditioning and prog. eqs

$$[\![\texttt{e:real}]\!] \in M(\mathbb{R})$$

- M should model prob. computations.
- M should validate equations from statistics.
- M should be commutative.
- Difficult to find such M due to conditioning.

Challenge 3: Conditioning and prog. eqs

$[\![\texttt{e:real}]\!] \in M(\mathbb{R})$ nonfinite measures

- M should model prob. computations.
- M should validate equations from statistics.
- M should be commutative.
- Difficult to find such M due to conditioning.

Challenge 3: Conditioning and prog. eqs

nearly-finite measures

$[\![\, \texttt{e:real} \,]\!] \in M(\mathbb{R})$ nonfinite measures

- M should model prob. computations.
- M should validate equations from statistics.
- M should be commutative.
- Difficult to find such M due to conditioning.

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```
f)
```

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```

~~[s b])~~

```
f)
```

Quasi-Borel space (QBS)

1. Continuous distributions.
2. Higher-order functions.
3. Conditioning and prog. eqs.

Big picture 1:
Extend measure theory
using category theory.

1. Continuous distr.
2. Higher-order fns.
3. Conditioning, prog. eqs.

~~1. Continuous distr.~~

2. Higher-order fns.

3. Conditioning, prog. eqs.

Meas_B

~~1. Continuous distr.~~

~~2. Higher-order fns.~~

3. Conditioning, prog. eqs.

Meas_B

Yoneda embedding ↘

$[\mathrm{Meas}_B{}^{op}, \mathrm{Set}]_\Pi$

~~1. Continuous distr.~~

~~2. Higher-order fns.~~

3. Conditioning, prog. eqs.

Meas_B

Yoneda embedding

Preserves nearly all the structures

$[\mathrm{Meas}_B{}^{op}, \mathrm{Set}]_\Pi$

~~1. Continuous distr.~~

~~2. Higher-order fns.~~

3. Conditioning, prog. eqs.

Meas_B

Yoneda embedding ↘

$[\mathrm{Meas}_B{}^{op}, \mathrm{Set}]_\Pi$

Enough structure for function types

~~1. Continuous distr.~~

~~2. Higher-order fns.~~

3. Conditioning, prog. eqs.

Meas_B

Yoneda embedding ↘

$[\mathrm{Meas}_B{}^{op}, \mathrm{Set}]_\Pi$

~~1. Continuous distr.~~

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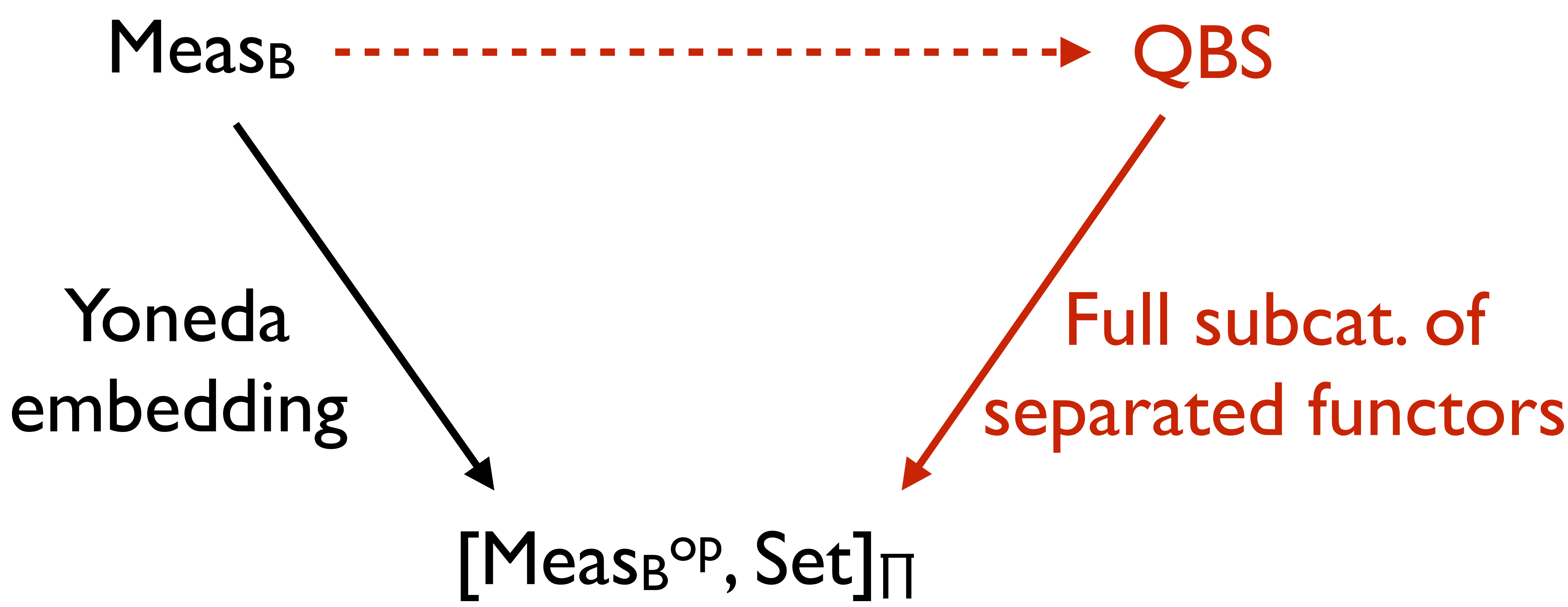
3. Conditioning, prog. eqs.



~~1. Continuous distr.~~

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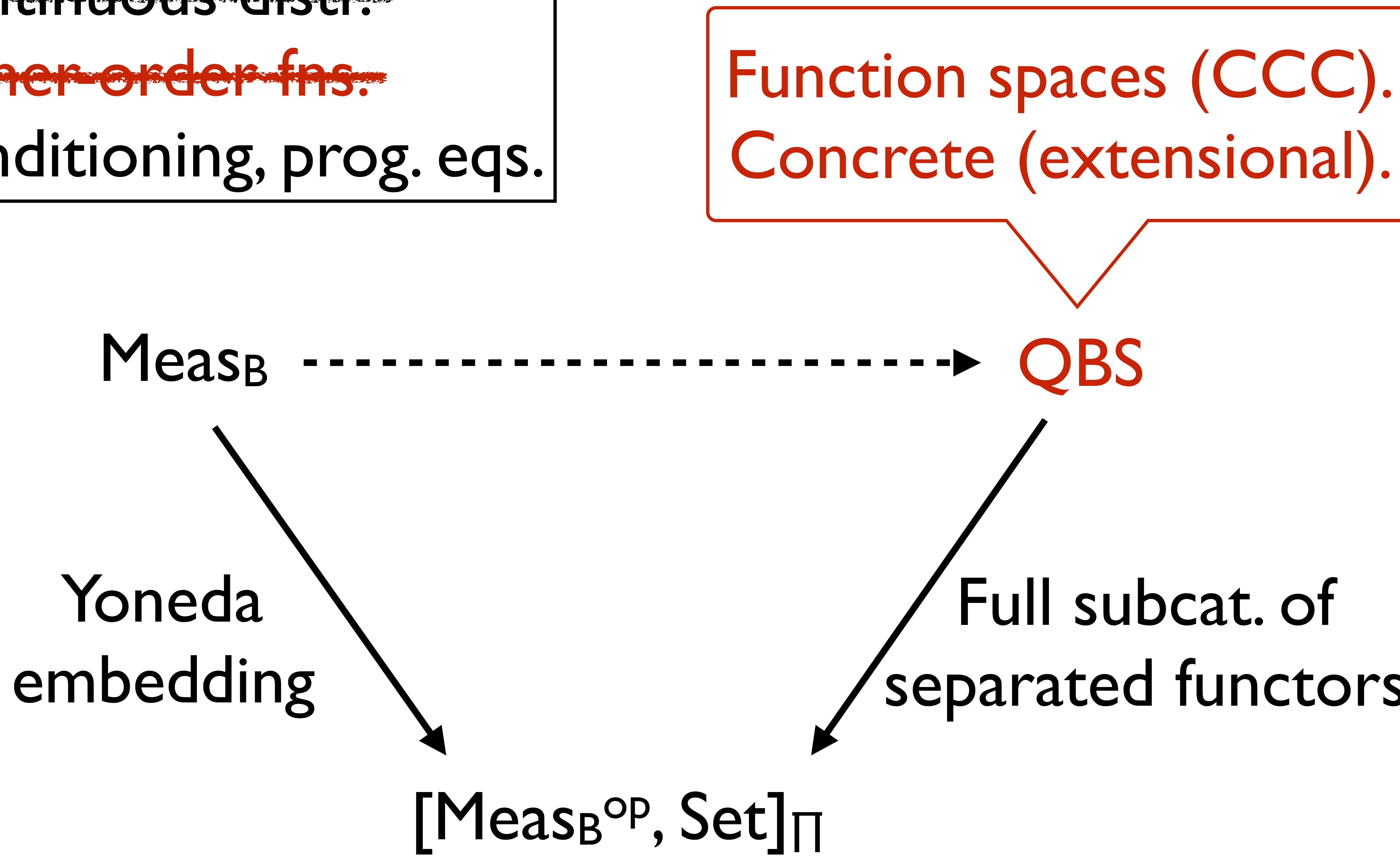
3. Conditioning, prog. eqs.



~~1. Continuous distr.~~
~~2. Higher-order fns.~~
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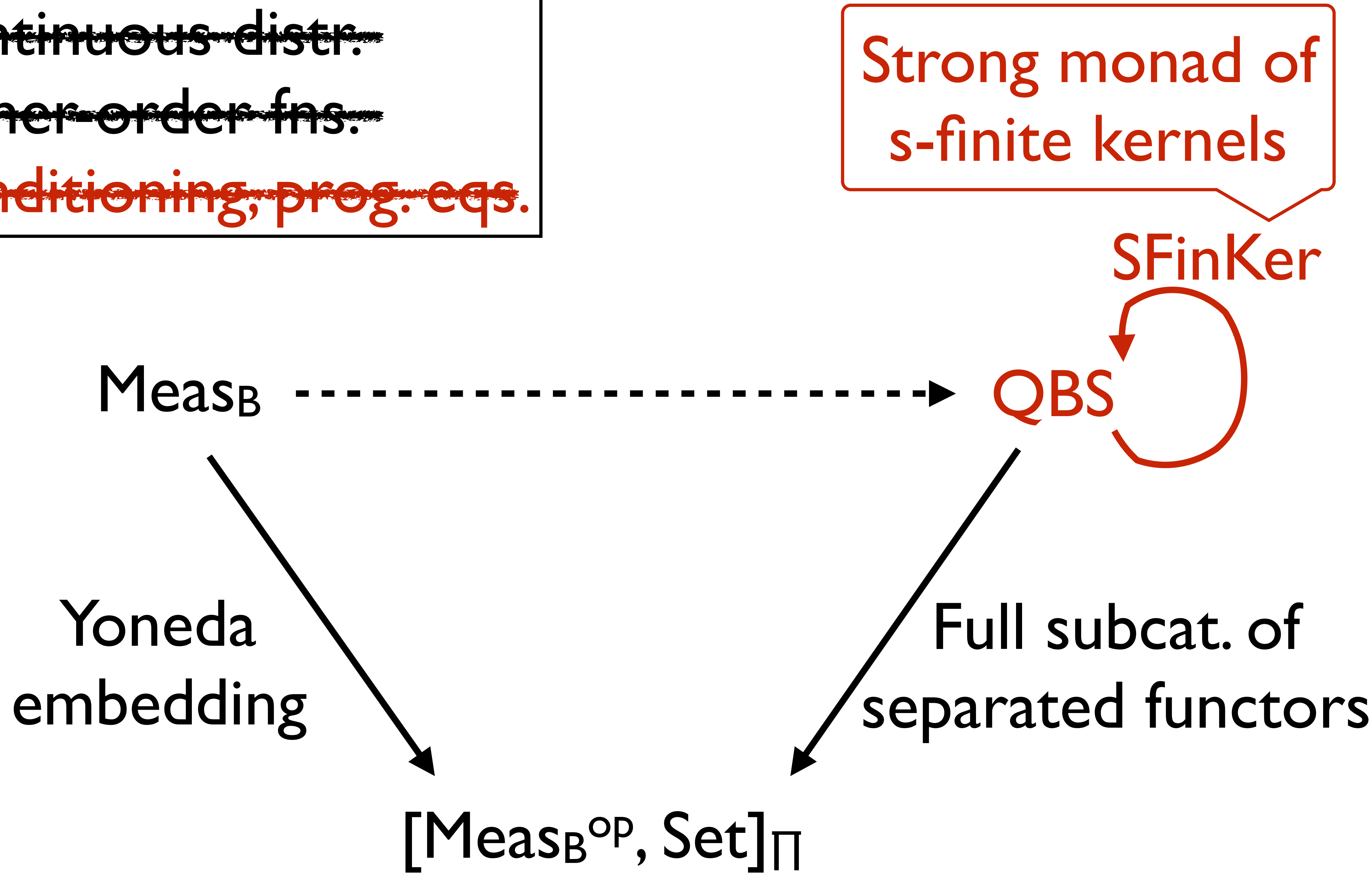

~~1. Continuous distr.~~
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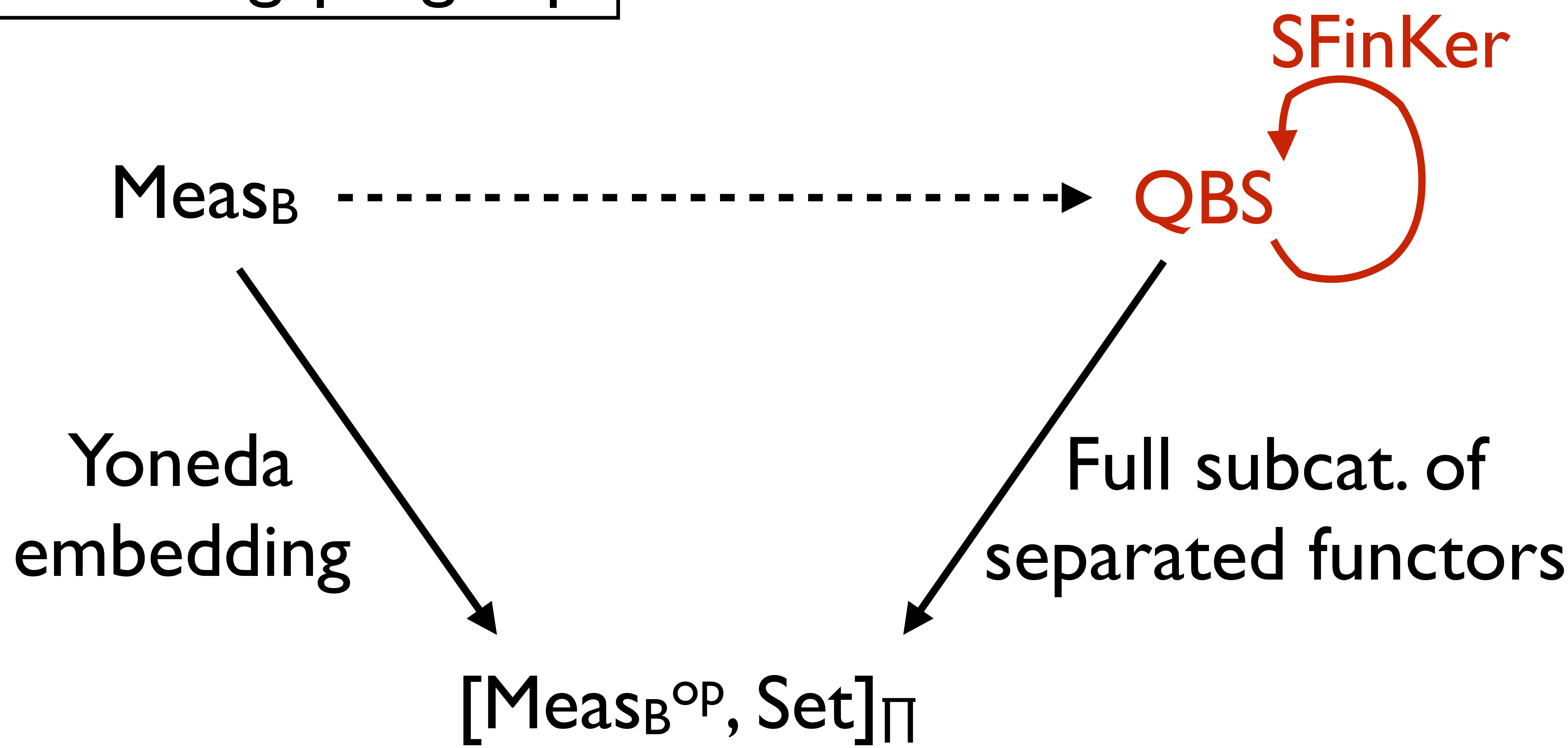

Big picture 2:
Random element first.

Random element α in X

Random element α in X

$$\alpha : \Omega \rightarrow X$$

- X - set of values.
- Ω - set of random seeds.
- Random seed generator.

Random element α in X in measure theory

$$\alpha : \Omega \to X$$

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Random element α in X
in measure theory

$$\alpha : \Omega \to X$$

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1. $\Sigma \subseteq 2^{\Omega}$, $\Theta \subseteq 2^{X}$

Random element α in X
in measure theory

$$\alpha : \Omega \rightarrow X$$

- X - set of values.
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- Random seed generator.

1. $\Sigma \subseteq 2^{\Omega}$, $\Theta \subseteq 2^{X}$
2. $\mu : \Sigma \rightarrow [0,1]$

Random element α in X
in measure theory

$\alpha : \Omega \to X$ is a random element
if $\alpha^{-1}(A) \in \Sigma$ for all $A \in \Theta$

- X - set of values.
- Ω - set of random seeds.
- Random seed generator.

1. $\Sigma \subseteq 2^{\Omega}$, $\Theta \subseteq 2^{X}$
2. $\mu : \Sigma \to [0,1]$

Random element α in X

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- X - set of values.
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Random element α in X in quasi-Borel spaces

$$\alpha : \Omega \to X$$

- X - set of values.
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Random element α in X in quasi-Borel spaces

$$\alpha : \mathbb{R} \to X$$

- X - set of values.
- $\mathbb{R}$ - set of random seeds.
- Random seed generator.

1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$

Random element α in X
in quasi-Borel spaces

$$\alpha : \mathbb{R} \to X$$

- X - set of values.
- $\mathbb{R}$ - set of random seeds.
- Random seed generator.

1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$

Random element α in X in quasi-Borel spaces

$$\alpha : \mathbb{R} \to X$$

- X - set of values.
- $\mathbb{R}$ - set of random seeds.
- Random seed generator.

1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$
3. $M \subseteq [\mathbb{R} \to X]$

Random element α in X
in quasi-Borel spaces

$\alpha : \mathbb{R} \to X$ is a random variable
if $\alpha \in M$

- X - set of values.
- $\mathbb{R}$ - set of random seeds.
- Random seed generator.

1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$
3. $M \subseteq [\mathbb{R} \to X]$

- Measure theory:
  - Measurable space $(X, \Theta \subseteq 2^X)$.
  - Random element is an induced concept.
- QBS:
  - Quasi-Borel space $(X, M \subseteq [\mathbb{R} \to X])$.
  - M is the set of random elements.

Rest of this tutorial

1. Baby measure theory.
   PL with cont. distribution.
2. Quasi-Borel space (QBS).
   PL with cont. distr. & HO fns.
3. SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.

Rest of this tutorial

1. Baby measure theory.
   PL with cont. distribution.
2. Quasi-Borel space (QBS).
   PL with cont. distr. & HO fns.
3. SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.

Programming language

- Will be sloppy about its syntax.
- Higher-order call-by-value probabilistic PL.

```
t  ::=  bool  |  real  |  t x t | t → t

e  ::=  ...
```

Baby measure theory

How to specify prob. μ?

How to specify prob. μ?

$X = \{0, 1, 2\}$.

Define $\mu : X \to [0,1]$. E.g., $\mu = [0.4, 0.4, 0.2]$.

Lifted $\mu : 2^X \to [0,1]$ by $\mu(A) = \sum_{x \in A} \mu(x)$.

How to specify prob. μ?

$X = \mathbb{R}$.

Define $\mu : X \to [0,1]$.

Lifted $\mu : 2^X \to [0,1]$ by $\mu(A) = \sum_{x \in A} \mu(x)$.

How to specify prob. μ?

$X = \mathbb{R}$.

Define $\mu : X \to [0,1]$.

Lifted $\mu : 2^X \to [0,1]$ by $\mu(A) = \sum_{x \in A} \mu(x)$.

Uncountable sum.
Typically ∞.

How to specify prob. μ?

$X = \mathbb{R}$.

~~Define $\mu : X \to [0,1]$.~~

~~Lifted~~ $\mu : 2^X \to [0,1]$ by ~~$\mu(A) = \sum_{x \in A} \mu(x)$.~~

Define

How to specify prob. μ?

X = $\mathbb{R}$.

~~Define μ : X→[0,1]~~

~~Lifted μ : 2^X→[0,1] by μ(A) = $\sum_{x \in A}$ μ(x).~~

~~Define~~

Pick a good collection $\Sigma \subseteq 2^X$.

Define μ : Σ→[0,1] with some care.

How to specify prob. μ?

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~~Define $\mu : X \to [0,1]$~~

Lifted ~~$\mu : 2^X \to [0,1]$ by $\mu(A) = \sum_{x \in A} \mu(x)$.~~

~~Define~~

σ-algebra

Pick a good collection $\Sigma \subseteq 2^X$.

Define μ : $\Sigma \to [0,1]$ with some care.

probability measure

Let $\Sigma \subseteq 2^X$.

Σ is a σ-algebra if it contains X, and is closed under countable union and set subtraction.

(X, Σ) is a measurable space if Σ is a σ-algebra.

Let $\Sigma \subseteq 2^X$.

Σ is a <u>σ-algebra</u> if it contains X, and is closed under countable union and set subtraction.

(X, Σ) is a <u>measurable space</u> if Σ is a σ-algebra.

$\mu : \Sigma \to [0,1]$ is a <u>probability measure</u> if $\mu(X)=1$ and $\mu(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ for all disjoint A_n's.

(X,Σ,μ) is a <u>probability space</u> if …

[Q] What are not measurable spaces?

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B}\text{x}\mathbb{B}, \{ A\text{x}B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \})$.
3. $(\mathbb{R}, \{ A \subseteq \mathbb{R} \mid A \text{ or } (\mathbb{R}\text{-}A) \text{ countable} \})$.
4. $(\mathbb{R}, \{ (r,s] \mid r<s \})$.

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[Q] Convert them to measurable spaces.

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2. $(\mathbb{B} \times \mathbb{B}, \{ A \times B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \})$.
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2. $(\mathbb{B}\text{x}\mathbb{B}, \sigma\{\, A\text{x}B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \,\})$.
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4. $(\mathbb{R}, \sigma\{\, (r,s] \mid r<s \,\})$.

Closure exists.
$\sigma(\Pi)$ smallest σ-algebra containing Π.

(X, Σ), (Y, Θ) - mBle spaces.

Product σ-algebra: $\Sigma \otimes \Theta = \sigma\{A \times B \mid A \in \Sigma, B \in \Theta\}$.

Product space: $(X,\Sigma) \times_m (Y, \Theta) = (X \times Y, \Sigma \otimes \Theta)$.

Borel σ-algebra on $\mathbb{R}$: $\mathfrak{B} = \sigma\{(r,s] \mid r < s\}$.

Borel space: $(\mathbb{R}, \mathfrak{B})$.

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$f:X\to Y$ is measurable (denoted $f:X\to_m Y$) if $f^{-1}(A)\in\Sigma$ for all $A\in\Theta$.

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$f: X \to Y$ is <u>measurable</u> (denoted $f: X \to_m Y$) if $f^{-1}(A) \in \Sigma$ for all $A \in \Theta$.

$k: X \times \Theta \to [0,1]$ is a <u>prob. kernel</u> if $k(x,-)$ is a prob. measure and $k(-,A)$ is measurable for all x, A.

Terms mean prob. kernels

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a prob. kernel from $[\![\Gamma]\!]$ to $[\![\mathtt{t}]\!]$.

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$= \int_A$ density-norm(s | r, 1) ds.

Rest of this tutorial

1. Baby measure theory.
   PL with cont. distribution.
2. Quasi-Borel space (QBS).
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   PL with cont. distr., HO fns & conditioning.

Quasi-Borel space

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such that M has enough random elements.

1. M contains all constant functions.
2. $(\alpha \circ \beta) \in M$ for all $\alpha \in M$ and mBle $\beta : \mathbb{R} \to \mathbb{R}$.
3. If $\mathbb{R} = \uplus_{i \in \mathbb{N}} R_i$ with $R_i \in \mathfrak{B}$ and $\alpha_1, \alpha_2, \ldots \in M$, then $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}} \in M$.

   Here $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}}(r) = \alpha_i(r)$ for all $r \in R_i$.

[Q] Pick a non-QBS.

1. $(\mathbb{R}, \{\alpha:\mathbb{R}\to\mathbb{R} \mid \alpha \text{ is a constant function}\})$.
2. $(\mathbb{R}, [\mathbb{R}\to\mathbb{R}])$.
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Standard way of converting a set to a QBS.

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3. ($\mathbb{R}$, $\{\alpha:\mathbb{R}\to\mathbb{R} \mid \alpha$ measurable wrt. $\mathfrak{B}\}$).

Standard way of converting a mBle space to a QBS.

(QBS) morphism

(X,M), (Y,N) - QBSes.

$f : X \to Y$ is a morphism if $(f \circ \alpha) \in N$ for all $\alpha \in M$.

Maps random elements to random elements.

We will write $f : X \to_q Y$.

[Th] QBSes form a cartesian closed category. So, they provide good product and function spaces.

[Q] What are the sets of random elements?

1. Product: $(X,M) \times_q (Y,N) = (Z,O)$.
   - $Z = X \times Y$, $\pi_1(x,y) = x$, $\pi_2(x,y) = y$.
   - $O = ???$
2. Fn space: $[(X,M) \to_q (Y,N)] = (Z,O)$
   - $Z = \{ f \mid f : X \to_q Y \}$, $ev(f,x) = f(x)$
   - $O = ???$

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1. Product: $(X,M) \times_q (Y,N) = (Z,O)$.
   - $Z = X \times Y$, $\pi_1(x,y) = x$, $\pi_2(x,y) = y$.
   - $O = \{ \langle\alpha,\beta\rangle \mid \alpha \in M \text{ and } \beta \in N \}$.
2. Fn space: $[(X,M) \rightarrow_q (Y,N)] = (Z,O)$
   - $Z = \{ f \mid f : X \rightarrow_q Y \}$, $ev(f,x) = f(x)$
   - $O = \{ curry(g) \mid g : \mathbb{R} \times_q X \rightarrow_q Y \}$.

Why works?

[NO] ev : $(\mathbb{R} \to_m \mathbb{R}) \times_m \mathbb{R} \to_m \mathbb{R}$

vs

[YES] ev : $(\mathbb{R} \to_q \mathbb{R}) \times_q \mathbb{R} \to_q \mathbb{R}$

Why works?

[NO] ev : $(\mathbb{R} \to_m \mathbb{R}) \times_m \mathbb{R} \to_m \mathbb{R}$

vs

[YES] ev : $(\mathbb{R} \to_q \mathbb{R}) \times_q \mathbb{R} \to_q \mathbb{R}$

Because the QBS product is more permissive.

Types mean QBSes

$$[\![\texttt{bool}]\!] = \text{MStoQBS}(\mathbb{B}, 2^{\mathbb{B}})$$

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$$[\![t \to t']\!] = [[\![t]\!] \to_q \text{Monad}([\![t']\!])]$$

$$[\![x_1{:}t_1, \ldots, x_n{:}t_n]\!] = [\![t_1]\!] \times_q \ldots \times_q [\![t_n]\!]$$

Terms mean morphisms almost

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a morphism $[\![\Gamma]\!] \to_{\mathtt{q}}$ Monad$[\![\mathtt{t}]\!]$.

Probability measure on Quasi-Borel space

A probability measure on a QBS (X,M) is a pair (α,μ) of $\alpha \in M$ and a prob. measure μ on $(\mathbb{R},\mathfrak{B})$.

A <u>probability measure</u> on a QBS (X,M) is a pair (α,μ) of $\alpha \in M$ and a prob. measure μ on $(\mathbb{R},\mathfrak{B})$.

random seed generator

A <u>probability measure</u> on a QBS (X,M) is a pair (α,μ) of $\alpha \in M$ and a prob. measure μ on $(\mathbb{R},\mathfrak{B})$.

seed convertor

A <u>probability measure</u> on a QBS (X,M) is a pair (α,μ) of α∈M and a prob. measure μ on ($\mathbb{R}$,$\mathfrak{B}$).

A <u>probability measure</u> on a QBS (X,M) is a pair (α,μ) of $\alpha \in M$ and a prob. measure μ on ($\mathbb{R}$,$\mathfrak{B}$).

E.g.

$$(X,M) = \text{MStoQBS}(\mathbb{B}, 2^{\mathbb{B}})$$

$$\mu = \text{uniform}(0,1], \quad \alpha(r) = \text{if } (r < 0.5) \text{ true false}$$

A <u>probability measure</u> on a QBS (X,M) is a pair (α,μ) of $\alpha \in M$ and a prob. measure μ on $(\mathbb{R}, \mathfrak{B})$.

E.g.

$$(X,M) = \mathrm{MStoQBS}(\mathbb{B}, 2^{\mathbb{B}})$$

μ = uniform(0,1], α(r) = if (r < 0.5) true false

μ' = uniform(0,2]/2, α'(r) = if (r < 1) true false

A probability measure on a QBS (X,M) is a pair (α,μ) of $\alpha \in M$ and a prob. measure μ on $(\mathbb{R},\mathfrak{B})$.

Quotient prob. measures by the smallest ~ s.t.

$$(\alpha,\mu) \sim (\beta,\nu)$$

if $\alpha \circ f = \beta$ and $\nu \circ f^{-1} = \mu$ for some $f:\mathbb{R} \to_m \mathbb{R}$.

$[\alpha,\mu]$ - equivalence class.

QBS of prob. measures

$$\mathrm{Prob}(X,M) = (Y,N)$$

Y = { [α,μ] | (α,μ) is a prob. meas. on (X,M) }.

N = { λr.[α,k(r)] | α∈M and k : $\mathbb{R}$x$\mathfrak{B}$→[0,1] is a prob. kernel }.

[Lem] Prob is a strong monad.

Completing the definitions

$$[\![t \to t']\!] = [[\![t]\!] \to_q \text{Prob}([\![t']\!])]$$

$[\![\Gamma \vdash e : t]\!]$ is a morphism $[\![\Gamma]\!] \to_q \text{Prob}[\![t]\!]$

Rest of this tutorial

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Conditioning and SFinKer monad on QBS

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Bayes's rule:

$$p(o \mid h) \times p(h) = p(h \mid o) \times p(o)$$

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$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a morphism $[\![\Gamma]\!] \rightarrow_q \text{Monad}[\![\mathtt{t}]\!]$

Bayes's rule:

p(o | h) x p(h) = p(h | o) x p(o)

1. Monad(_) = Prob(_). Sometimes undefined.

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a morphism $[\![\Gamma]\!] \to_q \text{Monad}[\![\mathtt{t}]\!]$

Bayes's rule:

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1. Monad(_) = Prob(_). Sometimes undefined.

2. Monad(_) = Prob([0,∞) $\times_q$ _).

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a morphism $[\![\Gamma]\!] \to_q \mathrm{Monad}[\![\mathtt{t}]\!]$

Bayes's rule:

$p(o \mid h) \times p(h) = p(h \mid o) \times p(o)$

1. Monad(_) = Prob(_). Sometimes undefined.
2. Monad(_) = Prob([0,∞) $\times_q$ _). Failed eqs.

Failed conjugate-prior equation from statistics

⟦
```
let x=sample(beta(1,1)) in
observe(flip(x),true);
x
```
⟧

$$\neq$$

⟦
```
observe(flip(0.5),true);
sample(beta(2,1))
```
⟧

$[\![\Gamma \vdash \texttt{e} : \texttt{t}]\!]$ is a morphism $[\![\Gamma]\!] \to_q \text{Monad}[\![\texttt{t}]\!]$

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1. Monad(_) = Prob(_). Sometimes undefined.
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$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a morphism $[\![\Gamma]\!] \to_q \mathrm{Monad}[\![\mathtt{t}]\!]$

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1. Monad(_) = Prob(_). Sometimes undefined.
2. Monad(_) = Prob($[0,\infty) \times_q$ _). Failed eqs.
3. Monad(_) = Meas(_). No commutativity.

Failed commutativity

$$[\![\ \texttt{let x=e in let y=e' in e''}\ ]\!] \neq [\![\ \texttt{let y=e' in let x=e in e''}\ ]\!]$$

if x doesn't occur in e' and y doesn't occur in e

$[\![\Gamma \vdash e : t]\!]$ is a morphism $[\![\Gamma]\!] \to_q \text{Monad}[\![t]\!]$

Bayes's rule:

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1. Monad(_) = Prob(_). Sometimes undefined.
2. Monad(_) = Prob($[0,\infty) \times_q$ _). Failed eqs.
3. Monad(_) = Meas(_). No commutativity.
4. Monad(_) = SFinKer(_).

QBS of prob. measures

$$\mathrm{Prob}(X,M) = (Y,N)$$

$$Y = \{\ [\alpha,\mu]\ \mid\ \alpha \in M,\quad \mu \text{ prob. meas. on } (\mathbb{R},\mathfrak{B})\ \}.$$

$$N = \{\ \lambda r.[\alpha,k(r)]\ \mid$$
$$\alpha \in M,\quad k : \mathbb{R} \times \mathfrak{B} \to [0,1] \text{ prob. kernel }\}.$$

QBS of ~~prob. measures~~ s-finite kernels

SFinKer(X,M)
~~Prob(X,M)~~ = (Y,N)

Y = { [α,μ] | α∈M, μ prob. meas. on ($\mathbb{R}$,$\mathfrak{B}$) }.

N = { λr.[α,k(r)] |
α∈M, k : $\mathbb{R}$x$\mathfrak{B}$→[0,1] prob. kernel }.

QBS of ~~prob. measures~~ s-finite kernels

SFinKer(X,M)
~~Prob(X,M)~~ = (Y,N)

Y = { [α,μ] | α∈M, μ ~~prob. meas.~~ s-finite measure on $(\mathbb{R},\mathfrak{B})$ }.

N = { λr.[α,k(r)] |
α∈M, k : $\mathbb{R}$x$\mathfrak{B}$→[0,1] prob. kernel }.

μ <u>finite</u> if like prob. measure but just $\mu(\mathbb{R})<\infty$.
μ <u>s-finite</u> if countable sum of finite measures.

QBS of prob. measures

k finite if like prob. kernel but $\sup_r k(r,\mathbb{R}) < \infty$.
k s-finite if countable sum of finite kernels.

SFinKer(X,M)
~~Prob(X,M)~~ = (Y,N)

$Y = \{ [\alpha,\mu] \mid \alpha \in M,$ μ ~~prob. meas.~~ on $(\mathbb{R},\mathfrak{B})$ }.
s-finite measure

$N = \{ \lambda r.[\alpha,k(r)] \mid$
$\alpha \in M,$ ~~$k : \mathbb{R} \times \mathfrak{B} \to [0,1]$ prob. kernel~~ }.
$k : \mathbb{R} \times \mathfrak{B} \to [0,\infty]$ s-finite kernel

μ finite if like prob. measure but just $\mu(\mathbb{R}) < \infty$.
μ s-finite if countable sum of finite measures.

[Th] SFinKer can be used to define the semantics of prob. PL with conditioning. (i.e., strong monad.)

[Th] It validates commutativity of programs and prog. eqs from statistics.

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$$[\![t \times t']\!] = [\![t]\!] \times_q [\![t']\!]$$

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$$[\![x_1{:}t_1, \ldots, x_n{:}t_n]\!] = [\![t_1]\!] \times_q \ldots \times_q [\![t_n]\!]$$

$[\![\Gamma \vdash \texttt{e} : \texttt{t}]\!]$ is a morphism $[\![\Gamma]\!] \to_q \mathrm{SFinKer}[\![\texttt{t}]\!]$

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References

1. A convenient category for higher-order probability theory. Heunen et a. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.

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